THERMAL PROCESSES IN SEMIMOMENT ASYMMETRIC HYDRODYNAMICS AND THE METHOD OF CHARACTERISTICS

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A resolving system of equations of the asymmetric theory of viscous fluid media with account for the propagation of thermal processes in it is derived under the condition of equality of the rotation of the local trihedron to the mean rotation of the field of permutation, and the equation of propagation of characteristics is obtained.

The asymmetric theory of viscous fluid media in which the rotation of the local trihedron is equal to the mean rotation of the field of permutation is considered:

$$\vec{\omega} = \frac{1}{2} \operatorname{rot} \vec{v}; \quad \vec{\omega} = (\omega_1, \omega_2, \omega_3) , \quad \vec{v} = (v_1, v_2, v_3) . \tag{1}$$

The equations of motion can be written in the form [1-3]

$$\sum_{j=1}^{3} \sigma_{ji,j} + X_i = \rho \frac{dv_i}{dt} , \quad \sum_{j=1}^{3} \mu_{ji,j} + \sum_{j,k=1}^{3} \varepsilon_{ijk} \sigma_{jk} + Y_i = j \frac{d\omega_i}{dt} . \tag{2}$$

Here the components of the force and moment stresses form nonsymmetric tensors, with

$$\sigma_{ij} = \mu (\partial_i v_j + \partial_j v_i) + \alpha (\partial_j v_i - \partial_i v_j) - 2\alpha \sum_{k=1}^{3} \varepsilon_{ijk} \omega_k + \left(\lambda \sum_{k=1}^{3} \partial_k v_k - p \right) \delta_{ij} ,$$

$$\mu_{ij} = \gamma (\partial_j \omega_i + \partial_i \omega_j) + \beta (\partial_j \omega_i - \partial_i \omega_j) .$$

From the second equation of system (2) we find the nonsymmetric part of the tensor of force stresses

$$\sigma_{\langle lm \rangle} \equiv \frac{1}{2} (\sigma_{lm} - \sigma_{ml}) = - \frac{1}{2} \sum_{i,j=1,3}^{3} \varepsilon_{ilm} \mu_{ji,j} - \frac{1}{2} \sum_{i=1}^{3} \varepsilon_{ilm} \left(Y_i - J \frac{d\omega_i}{dt} \right) .$$

Taking into consideration that $\sigma_{(ij)} = \frac{1}{2} (\sigma_{ij} + \sigma_{ji})$, we have

$$\sigma_{(ij)} = \mu (\partial_j v_i + \partial_i v_j) + \left(\lambda \sum_{k=1}^{3} \partial_k v_k - p \right) \delta_{ij} ,$$

$$\sigma_{\langle ij \rangle} = \alpha (\partial_j v_i - \partial_i v_j) - 2\alpha \sum_{k=1}^{3} \varepsilon_{ijk} \omega_k .$$

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Upon substitution into the first equation of system (2), we obtain

$$\mu\Delta\vec{v} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{v} - \mathrm{grad}\,p + \frac{1}{4}(\gamma+\beta)\,\mathrm{rot}\,\mathrm{rot}\,\Delta\vec{v} +$$

$$+\frac{1}{2}\,\mathrm{rot}\left(\vec{Y} - j\frac{d\vec{\omega}}{dt}\right) + \vec{X} = \rho\frac{d\vec{v}}{dt}$$

or, following (1), we have

$$\mu\Delta\vec{v} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{v} - \mathrm{grad}\,p + \frac{1}{4}(\gamma+\beta)\,\mathrm{rot}\,\mathrm{rot}\,\Delta\vec{v} +$$

$$+\frac{1}{2}\,\mathrm{rot}\,\vec{Y} + \vec{X} = \rho\frac{d\vec{v}}{dt} + \frac{j}{4}\,\mathrm{rot}\left[\frac{d}{dt}(\mathrm{rot}\,\vec{v})\right]. \tag{3}$$

We add the continuity equation

$$\frac{d\rho}{dt} + p\,\mathrm{div}\,\vec{v} = 0\,. \tag{4}$$

to Eq. (3).

To describe thermal processes, we use the equation of heat conduction [3] in the form

$$\theta_{,jj} - \frac{1}{\chi}\frac{d\theta}{dt} - \eta\left(1 + \frac{\theta}{T_0}\right)\mathrm{div}\,\vec{v} = -\frac{Q}{\chi}\,, \tag{5}$$

and ν grad θ should be added to Eq. (3) since thermal processes in a fluid medium are accompanied by volumetric deformations.

In this form, system (3)-(5) is nonlinear and, to simplify it, we suppose that the motions occur with small velocities. This makes it possible to discard the nonlinear terms. The fluid is incompressible, and therefore we assume div $\vec{v} = 0$ and ρ = const.

The resolving system of equations takes the form

$$\begin{gathered}
\mu\Delta\vec{v} - \frac{1}{4}(\gamma+\beta)\,\Delta^2\vec{v} - \mathrm{grad}\,p + \frac{1}{2}\,\mathrm{rot}\,\vec{Y} + \vec{X} = \frac{\partial}{\partial t}\left(\rho\vec{v} - \frac{j}{4}\Delta\vec{v}\right) + \nu\,\mathrm{grad}\,\theta\,, \\
-\mathrm{div}\,\vec{v} = 0\,, \\
\Delta\theta - \frac{1}{\chi}\frac{\partial\theta}{\partial t} = -\frac{Q}{\chi}\,.
\end{gathered} \tag{6}$$

We introduce the operator notation

$$\Pi = \mu\Delta - \frac{1}{4}(\gamma+\beta)\,\Delta^2 + \frac{\partial}{\partial t}\left(\frac{j}{4}\Delta - \rho\right).$$

Then

$$\begin{gathered}
\Pi\vec{v} - \mathrm{grad}\,p - \nu\,\mathrm{grad}\,\theta = -\frac{1}{2}\,\mathrm{rot}\vec{Y} - \vec{X}, \\
-\mathrm{div}\,\vec{v} = 0\,, \\
\Delta\theta - \frac{1}{\chi}\frac{\partial\theta}{\partial t} = -\frac{Q}{\chi}\,.
\end{gathered} \tag{7}$$

To derive an equation of the characteristic surface of system (7), we change over to the new variables $x'_k = \Omega_k\,(x_1, ..., x_4)$, $k = \overline{1,4}$. We assume that $x_1 = t$, and x_2, x_3, x_4 are Cartesian coordinates. Then

$$\frac{\partial v_i}{\partial x_k} = \sum_{s=1}^{4} \frac{\partial v_i}{\partial x'_s}\frac{\partial \Omega_s}{\partial x_k}, \quad \frac{\partial^2 v_i}{\partial x_n \partial x_k} = \sum_{s=1}^{4} \frac{\partial v_i}{\partial x'_s}\frac{\partial^2 \Omega_s}{\partial x_n \partial x_k} + \dots ,$$

$$\frac{\partial^3 v_i}{\partial x_m \partial x_n \partial x_k} = \sum_{s=1}^{4} \frac{\partial v_i}{\partial x'_s}\frac{\partial^3 \Omega_s}{\partial x_m \partial x_n \partial x_k} + \dots ,$$

$$\frac{\partial^4 v_i}{\partial x_q \partial x_m \partial x_n \partial x_k} = \sum_{s=1}^{4} \frac{\partial v_i}{\partial x'_s}\frac{\partial^4 \Omega_s}{\partial x_q \partial x_m \partial x_n \partial x_k} + \dots .$$

We substitute these expressions into system (7) and write out only the terms that contain the derivatives $\partial v_i/\partial x'_1$ and $\partial\theta/\partial x'_1$. We find the equation of characteristics from the condition that (7) does not contain the derivatives $\partial v_i/\partial x'_1$ and $\partial\theta/\partial x'_1$, i.e., the determinant formed from the coefficients of these derivatives is zero. Therefore

$$\begin{vmatrix} \Pi\,\Omega & 0 & 0 & -\nu\dfrac{\partial\Omega}{\partial x_1} & -\dfrac{\partial\Omega}{\partial x_1} \\ 0 & \Pi\,\Omega & 0 & -\nu\dfrac{\partial\Omega}{\partial x_2} & -\dfrac{\partial\Omega}{\partial x_2} \\ 0 & 0 & \Pi\,\Omega & -\nu\dfrac{\partial\Omega}{\partial x_3} & -\dfrac{\partial\Omega}{\partial x_3} \\ 0 & 0 & 0 & \left(\Delta\Omega - \dfrac{1}{\chi}\dfrac{\partial\Omega}{\partial t}\right) & 0 \\ -\dfrac{\partial\Omega}{\partial x_1} & -\dfrac{\partial\Omega}{\partial x_2} & -\dfrac{\partial\Omega}{\partial x_3} & 0 & 0 \end{vmatrix} = 0 ,$$

whence we have the following equation of characteristics for system (7):

$$(\Pi\,\Omega)^2 g^2 \left(\Delta\Omega - \frac{1}{\chi}\frac{\partial\Omega}{\partial t}\right) = 0$$

or

$$\left[\mu\Delta\Omega - \frac{1}{4}(\gamma+\beta)\,\Delta^2\Omega + \frac{\partial}{\partial t}\left(\frac{j}{4}\Delta\Omega - \rho\Omega\right)\right]\left(\Delta\Omega - \frac{1}{\chi}\frac{\partial\Omega}{\partial t}\right) = 0 . \tag{8}$$

We obtain from (8)

$$\mu\Delta\Omega - \frac{1}{4}(\gamma+\beta)\,\Delta^2\Omega + \frac{\partial}{\partial t}\left(\frac{j}{4}\Delta\Omega - \rho\Omega\right) = 0 . \tag{9}$$

$$\Delta\Omega - \frac{1}{\chi}\frac{\partial\Omega}{\partial t} = 0 . \tag{10}$$

At $j = \mu = 0$ and $\gamma + \beta = 0$ Eq. (9) involves the case of stationary discontinuity [4].

NOTATION

α, λ, μ, γ, β, χ, η, ν, material constants of the fluid medium; p, pressure; X_i, Y_i, internal forces and distributed moments; j, moment of inertia; ε_{ijk}, Levi-Civita symbol; δ_{ij}, Kronecker symbol; σ, stress tensor; $\vec{v}$, linear velocity; $\vec{\omega}$, velocity of rotary motion; θ, increase in the temperature compared to the temperature of the natural state T_0; Q, heat supplied from the outside; ρ, medium density; $g^2 = \sum_{k=1}^{3} \{\frac{\partial\Omega}{\partial x_k}\}^2$; s, summation index.

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